Products of primes in weak systems of Arithmetic

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A simple case of MRDP theorem.

$$\psi(b,d) = \forall x \leq b \forall y \leq b(x^2 + y^2 - 2y + d = 0).$$

$p_+(x,y,d) = x^2 + y^2 + d$ and $p_-(x,y,d) = 2y$ so the maximum value of the term $p_+ + p_-$ is $2b^2 + d + 2b$. This is $\max(\psi)$.

The number of all possible pairs (x,y) is $(b+1)^2$

Let *pair* be any pairing function and assume there exist $(b+1)^2$ consecutive (relatively) prime numbers$>$ $\max(\psi)$:

$$p_0 = p_{pair(0,0)}, p_1 = p_{pair(0,1)}, p_2 = p_{pair(1,0)}, \cdots p_{pair(b,b)}$$

as well as their product *prod*.

Chinese Remainder Theorem. Suppose that there exists t_1, t_2 that *encode* all these pairs i.e.

$$\begin{aligned} t_j &\equiv 0 \pmod{p_0} \\ &\vdots \\ t_j &\equiv i_j \pmod{p_{pair(i_1,i_2)}} \\ &\vdots \\ t_j &\equiv b \pmod{p_{pair(b,b)}} \end{aligned}$$

for all $0 \leq i_1, i_2 \leq b$, $j = 1$ or $j = 2$.

So ψ is equivalent with
$\exists t_1 \exists t_2$(above systems $\wedge\, t_1^2 + t_2^2 - 2t_2 + d \equiv 0 \pmod{prod}$)
which is an $\exists_1$ formula.

Problem.

Given any $M \models I\Delta_0$ and $m \in M$, are there any consecutive primes $\geq m$ as well as their product? Can we enumerate them?

Definitions. 1) $primeprod(n, n') = \prod_{n \leq p \leq n' \wedge prime(p)} p$

2) The number of primes in the interval $[n, n']$ is m iff

$$\exists z (z = \prod_{i \leq n'} F_n(i) \wedge 2^m = z).$$

A. Berarducci, P. D' Aquino Δ_0- *complexity of the relation* $y = \prod_{i \leq n} F(i)$ Ann. Pure & App. Logic, 1995

$$F_n(i) = \begin{cases} 1, & \text{if } \neg prime(i) \vee i < n \\ 2, & \text{if } prime(i) \wedge i \geq n \end{cases}$$

We write $\pi_{\geq n}(n') = m$ iff the number of primes in the interval $[n, n']$ is m. As usual we define $\pi(n') := \pi_{\geq 2}(n')$.

3) Let $p \geq n$ a prime. p is the m-th prime $\geq n$ iff $\pi_{\geq n}(p) = m$.

Enumeration of primes costs:

$I\Delta_0 \vdash exp \leftrightarrow \forall l \exists p(prime(p) \wedge \pi_{\geq 2}(p) = l)$.

Ch. Cornaros and C. Dimitracopoulos: *A note on exponentiation*, J. Symbolic Logic 58 (1993), 64–71.

Classes of logarithmic powers.

$\log(x)$ denotes $[\log_2(x)]$, $\log^{(2)}(x)$ denotes $[\log_2([\log_2(x)])]$ etc.

4) Ω_n is defined for all $n \in \mathbb{N}$:

$$\forall x \forall y \exists z (x^{(\log y)^{(\log^{(2)} y)^{\cdot^{\cdot^{\cdot^{\log^{(n)} y}}}}}} = z)$$

5) Ω_n^* : $\forall x \forall y \exists z (x^{\log^{(n)} y} = z)$.

Some basic Lemmas.

A) If $p \geq n$ is a prime and $primeprod(n, p)$ exists, then there exists some m such that p is the m-th prime $\geq n$.

B) $I\Delta_0 \vdash \forall n \geq 2 \forall m \geq 2 \forall p[primeprod(n, p)$ exists $\wedge$ p is the $\log(m)$-th(concequtive) prime $\geq n \rightarrow n^{\log(m)}$exists$]$.

Corollary: If $M \models I\Delta_0$ has all the products of logarithmically many consecutive primes, then $M \models \Omega_1$.

Similar results hold for models of Ω_n or Ω_n^*.

Converse directions?

Bertrand's Postulate: $\forall x > 1 \exists p < 2x (prime(p) \wedge p > x)$.

$I\Delta_0$+Bertrand's Postulate
$\vdash \forall n \geq 2 \forall m \geq 2 (\exists p)(p \text{ is the } \log(m)\text{-th prime } \geq n)$.

but $I\Delta_0$+Bertrand's Postulate $\nvdash \exists$ products of logarithmically many consecutive primes above any number n.

..... We need some power!

Adding some Ω "power" in M

$I\Delta_0+$ Bertrand's Postulate $+\Omega_1 \vdash \exists$ products of logarithmically many consecutive primes above any number n.

Similar results: $I\Delta_0+$ Bertrand's Postulate $+\Omega_s^*$
$\vdash \forall n \forall m \geq 2^s \exists$ products of $\log^{(s)}(m)$ many consecutive primes above n.

$I\Delta_0+$ Bertrand's Postulate $+\Omega_s \vdash \forall n \forall m \geq 2^s \exists$ products of

$\log(m)^{\log^{(2)}(m)^{\cdot^{\cdot^{\cdot^{\log^{(s)}(m)}}}}}$ many consecutive primes above n.

A Question

Does the existence of products of logarithmically many consecutive primes above any number n guarantee some "local" versions of Bertrand's Postulate?

$I\Delta_0 + \exists$ product of $\log(n)$ many consecutive primes above $n \vdash \forall 1 < x \leq n^2 \exists p < 2x(prime(p) \wedge p > x)$?

How many primes ≥ 2 exist in $M \models I\Delta_0$?

Answer: If M is any $M \models I\Delta_0$ then there exist at least $\sqrt{\log(m)}$ many primes for any $m \geq 2$.

If M is any $M \models I\Delta_0$+Bertrand's Postulate, then for any $m \geq 2$ there are at least $\log(m)$ primes≥ 2 i.e.
$\exists p \leq m(prime(p) \wedge \pi(p) \geq \log(m))$.

$M \models I\Delta_0+$ weak-$PHP(\Delta_0)$ and m is an element of M such that $2^m \in M$, then there exist more than m primes≥ 2 in the interval $[2, 2^{6m}]$.

Weak PHP(Δ_0)

$\forall x \neg (f : x^2 \to x$ and f is 1-1) for any Δ_0 definable f.

$I\Delta_0 + \Omega_s^*, s \geq 1$, is strong enough to prove weak-$PHP(\Delta_0)$.
weak-$PHP(\Delta_0)$ is strong enough to prove $\pi_{\geq m}(m^{11}) > 0$ i.e. there are many primes above any $m \in M$. For details see

J. B. Paris, A. J. Wilkie and A. R. Woods: *Provability of the pigeonhole principle and the existence of infinitely many primes*, J. Symbolic Logic 53 (1988), 1235–1244.

Some Consequences

If M satisfies some Ω then M has strictly **more** primes than $\log(m), m > 1$.

Examples. If $M \models I\Delta_0 + \Omega_1$, then there exist at least $log^2(m)$ many primes for any $m > 1$ i.e. there exists some prime p: $\pi(p) \geq log^2(m)$.

If $M \models I\Delta_0 + \Omega_2^*$, then there are at least $log(m)log^{(2)}(m)$ many primes for any $m > 1$ i.e. there exists some prime p: $\pi(p) \geq \log(m)log^{(2)}(m)$.

Equivalences of Ω hierarchy with products of primes under Bertrand's Postulate.

If we **add** the hypothesis that $M \models$Bertrand's Postulate then we can also conclude that the product of these primes also exists:

$I\Delta_0+$Bertrand's Postulate $\vdash \forall m > 1 \exists p \exists x(x = primeprod(2, p)$ $\wedge p$ is the $\log^2(m)$-th prime $\geq 2) \leftrightarrow \Omega_1$. Bertrand's Postulate is needed only for the implication($\leftarrow$).

Similar equivalences for all Ω axioms. For example:
$I\Delta_0+$Bertrand's Postulate $\vdash \forall m > 1 \exists p \exists x(x = primeprod(2, p)$ $\wedge p$ is the $\log(m)^{\log^{(2)}(m)}$-th prime $\geq 2) \leftrightarrow \Omega_2$. Bertrand's Postulate is needed only for the implication($\leftarrow$).

Equivalences of Ω hierarchy with products of primes without Bertrand's Postulate.

Paola D' Aquino results:

$I\Delta_0 \vdash \forall x > 1 \forall y(y = 2^x \rightarrow \exists z < y^2(z = primeprod(2, x))$
$I\Delta_0 \vdash \forall x > 1 \forall z(z = primeprod(2, x) \rightarrow \exists y < \mu z^4(2^x = y))$, for some small $\mu \in \mathbb{N}$.

Paola D' Aquino Thesis *Exponentiation and Fragments of Arithmetic*, Oxford University 1992.

Useful Corollaries

A) Number of primes: $I\Delta_0 \vdash \forall s \geq 2[primeprod(2, s^5)$ exists $\rightarrow \exists p \leq s^5(p$ is the $[\frac{s}{2}]$-th prime$\geq 2)]$.

B) Exponentiation power:
$I\Delta_0 \vdash \forall m \geq 2 \forall z \leq m(primeprod(2, z \log(m))$ exists $\leftrightarrow 2^{z \log(m)}$ exists $\leftrightarrow m^z$ exists)

Replacing $z = log^{(n)}(m)$ for any natural $n \geq 1$ we also take
$I\Delta_0 \vdash \forall m \geq 2(primeprod(2, log^{(n)}(m) \log(m))$ exists $\leftrightarrow m^{log^{(n)}(m)}$ exists.

Useful Corollaries

Ω hierarchy:

$I\Delta_0 \vdash \Omega_1 \leftrightarrow \forall m \geq 2(primeprod(2, \log^2(m))$ exists)

$I\Delta_0 \vdash \Omega_s^* \leftrightarrow \forall m \geq 2^s(primeprod(2, \log^{(s)}(m) \log(m))$ exists)

$I\Delta_0 \vdash \Omega_s \leftrightarrow$

$\forall m \geq 2^s(primeprod(2, \log(m)^{\log^{(2)}(m)^{\cdot^{\cdot^{\cdot^{\log^{(s)}(m)}}}}} \log(m))$ exists)

$I\Delta_0 \vdash exp \leftrightarrow \forall m \geq 2primeprod(2, m)$ exists

Last Remark and a Problem

Note that *primeprod*$(2, p)$, where p is the $\log^2(m)$-th prime ≥ 2 is obviously bigger than *primeprod*$(2, p')$, where p' is the biggest prime $p' \leq \log^2(m)$. So the first below does not necessarily implies the second:

$I\Delta_0 \vdash \forall m \geq 2(\mathit{primeprod}(2, \log^2(m))$ exists$) \leftrightarrow \Omega_1$

($I\Delta_0+$Bertrand's Postulate $\vdash$)
$\forall m \geq 2\exists p\exists x(x = \mathit{primeprod}(2, p) \wedge p$ is the $\log^2(m)$-th prime $\geq 2) \leftrightarrow \Omega_1$

Problem: ? $I\Delta_0 + \Omega_1 \vdash \forall m \geq 2\exists p\exists x(x = \mathit{primeprod}(2, p) \wedge p$ is the $\log^2(m)$-th prime $\geq 2)$

Grazie!